AIAA JOURNAL
Vol. 48, No. 6, June 2010

Technical Notes

Energy Transformation to Generalized Timoshenko Form for Nonuniform Beams

Jimmy C. Ho* and Dewey H. Hodges†
Georgia Institute of Technology, Atlanta, Georgia 30332
and
Wenbin Yu‡
Utah State University, Logan, Utah 84322

DOI: 10.2514/1.J050160

Nomenclature

A, B, C, D = stiffness matrices of the second-order asymptotically correct beam theory
a = characteristic length of the cross-sectional dimension
$\mathbf{a_i}$ = unit vectors of the reference coordinate system
b = half-width of a beam with a rectangular cross section
E = Young's modulus
F_1 = internal axial force
F_2, F_3 = internal shear forces
k_1 = initial twist
k_2, k_3 = initial curvatures
ℓ = characteristic wavelength of deformation along x_1
M_1 = internal torque
M_2, M_3 = internal bending moments
$\mathbf{n}$ = outward-directed unit normal vector
R = characteristic radius of initial twist and curvatures
$\bar{R}, \bar{S}, \bar{T}$ = submatrices of the generalized Timoshenko flexibility matrix
t = thickness of a beam with a rectangular cross section
U = sectional strain energy
X, Y, G = submatrices of the generalized Timoshenko stiffness matrix
x_1 = beam axial coordinate
x_2, x_3 = local, Cartesian, cross-sectional coordinates for the beam section
γ_s = $\lfloor 2\gamma_{12} \quad 2\gamma_{13} \rfloor^T$, column matrix of transverse shear strain measures defined in generalized Timoshenko theory
Δ_i = identity matrix of order i
ϵ = $\lfloor \gamma_{11} \quad \kappa_1 \quad \kappa_2 \quad \kappa_3 \rfloor^T$, column matrix of a subset of strain measures defined in generalized Timoshenko theory
$\bar{\epsilon}$ = $\lfloor \bar{\gamma}_{11} \quad \bar{\kappa}_1 \quad \bar{\kappa}_2 \quad \bar{\kappa}_3 \rfloor^T$, column matrix of strain measures defined in classical beam theory
ν = Poisson's ratio
τ = local taper parameter of a linearly tapered beam

Superscript

$'$ = partial derivative with respect to x_1

I. Introduction

THREE-DIMENSIONAL (3-D) spanwise nonuniformity effects exist in the two-dimensional (2-D) cross-sectional analysis of nonuniform beams. This statement is easily justified by noting that stresses at boundaries that vary geometrically along the length violate the traction-stress boundary relationships from Cauchy's formula unless nonuniformity effects are properly modeled. In Fig. 1, a prismatic beam is shown alongside a linearly tapered beam. Unit vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ are parallel to x_1 and x_2, respectively. For the tapered beam, the slopes of the lower and upper edges are denoted by τ and $-\tau$, respectively; thus, in contrast to the prismatic case, $\mathbf{n}$ features a component parallel to $\mathbf{a}_1$ that must be accounted for if the predicted stresses are to be consistent with the surface boundary conditions. This component of $\mathbf{n}$ that is parallel to $\mathbf{a}_1$ is normal to the plane of the cross section, so $\mathbf{n}$ itself contributes to the 3-D nonuniformity effects. The strain energy is an integration, of the product between stresses and strains, over the 3-D domain of the structure. Elastic stiffness constants are determined by the sectional strain energy, so the fact that nonuniformity affects the stresses means that it also affects the stiffness constants. All current 2-D sectional analysis tools implicitly assume that $\mathbf{n}$ has no component that is parallel to x_1. Therefore, nonuniformity effects have been neglected thus far.

Since beam theories are derived from the theory of elasticity, exact elasticity solutions provide the standard by which the accuracy of beam theory may be assessed. Elasticity solutions are known to exist for nonuniform beams only in the case of a linearly tapered isotropic strip beam under specific loading conditions. The geometry of the tapered strip beam is already depicted in Fig. 1. It is assumed to be sufficiently thin that the governing elasticity equations are the 2-D plane-stress equations. Timoshenko and Goodier [1] provide solutions to the two cases, where loading is applied such that either the internal axial force or bending moment is constant along the length. Krahula [2] provides a solution for the flexure case, where loading is applied such that the internal shear force is constant along the length.

Hodges et al. [3] recently demonstrated success at incorporating nonuniformity effects using beam theory for the linearly tapered strip. An analytical solution of the cross-sectional analysis was obtained by the variational-asymptotic method (VAM) of Berdichevsky [4] and verified to be in excellent agreement with the exact elasticity solutions. For a section having $\nu = 0.3$ and a linear taper of $\tau = 0.1763$, which corresponds to 10° taper and is not uncommon as local taper on the transition regions of helicopter rotor blades, nonuniformity effects reduce the effective bending stiffness, from not featuring taper, by 4.28%. This decrease in bending stiffness is large enough that nonuniformity effects cannot be ignored. Note that the tapered strip analyzed is initially straight and untwisted.

A 2-D cross-sectional analysis tool that would accommodate sections of arbitrary materials and geometries must rely on a numerical approach. This is the approach taken by the commercial

Received 27 August 2009; revision received 22 December 2009; accepted for publication 5 February 2010.

*Currently Research Scientist, ELORET Corporation, U.S. Army Aeroflightdynamics Directorate, Ames Research Center, Mail Stop 215/1, Moffett Field, CA 94035; jimmy.c.ho@us.army.mil. Lifetime Senior Member AIAA.

†Professor, Guggenheim School of Aerospace Engineering, 270 Ferst Drive; dhodges@gatech.edu. Fellow AIAA.

‡Associate Professor, Department of Mechanical and Aerospace Engineering, 4130 Old Main Hill; wenbin@engineering.usu.edu. Lifetime Senior Member.

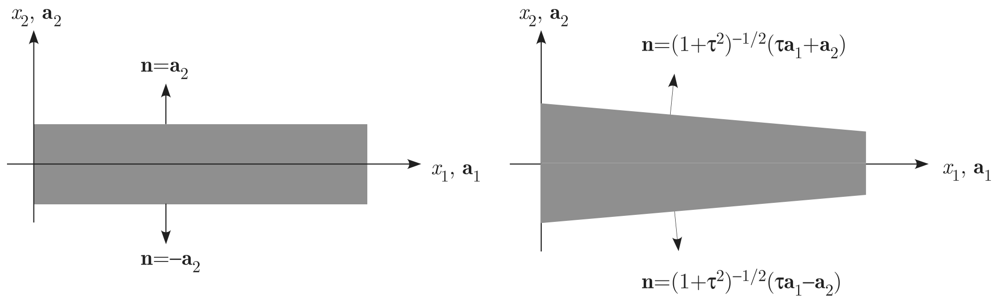

Fig. 1 Contrasting a prismatic beam with a tapered beam.

software Variational-Asymptotic Beam Sectional (VABS) Analysis [5], which discretizes the sectional warping deformation using 2-D finite elements. Theoretical foundations of VABS include decomposition of the rotation tensor [6] and the aforementioned VAM. To attain the elastic stiffness constants of generalized Timoshenko beam theory, VABS first solves for a second-order asymptotically correct sectional strain energy. It then transforms this strain energy into generalized Timoshenko form in order to obtain the stiffness constants. The solution procedure from [3] is an analytical analogue to that of VABS and also entails the energy transformation.

The purpose of this Technical Note is to present the transformation of the strain energy into generalized Timoshenko form, while accounting for nonuniformity effects, for use with analyzing arbitrary sections. What is presented here allows for the possibility that the beam may feature variations in initial twist and initial curvatures before loading. As validation, results from the present numerical approach (for arbitrary sections) are shown and compared against the analytical solutions of [3] for the linearly tapered isotropic strip.

II. Energy Transformation

The energy transformation is from that of a second-order asymptotically correct beam theory to that of the generalized Timoshenko beam theory. For prismatic beams, the transformation procedure is first presented by Popescu and Hodges [7]. For uniform beams featuring constant initial twist and/or curvatures, the procedure is presented by Yu et al. [8].

The form of two times the second-order asymptotically correct sectional strain energy, as derived from the VAM, is

$$2U = \bar\epsilon^T A\bar\epsilon + 2\bar\epsilon^T B\bar\epsilon' + \bar\epsilon'^T C\bar\epsilon' + 2\bar\epsilon^T D\bar\epsilon'' \tag{1}$$

The one-dimensional (1-D) generalized strain measures of classical Euler–Bernoulli beam theory are $\bar\epsilon = \lfloor \bar\gamma_{11} \quad \bar\kappa_1 \quad \bar\kappa_2 \quad \bar\kappa_3 \rfloor^T$ where $\bar\gamma_{11}$ is the extensional strain measure, $\bar\kappa_1$ is the torsional strain measure, $\bar\kappa_2$ and $\bar\kappa_3$ are bending strain measures. Matrices A, B, C, and D carry both sectional geometry and material information. Explicit expressions for A, B, C, and D, which include nonuniformity effects, are given by Ho et al. [9]. Nonuniformity effects appear in these matrices as derivatives of the warping field with respect to x_1. Unfortunately, the terms that represent nonuniformity effects in A and B have not been successfully validated by Ho [10]. The focus of this Technical Note is on the energy transformation, so it is assumed here that A, B, C, and D are known.

The form of two times the generalized Timoshenko sectional strain energy is

$$2U = \epsilon^T X\epsilon + 2\epsilon^T Y\gamma_s + \gamma_s^T G\gamma_s \tag{2}$$

The strain measures are now $\epsilon = \lfloor \gamma_{11} \quad \kappa_1 \quad \kappa_2 \quad \kappa_3 \rfloor^T$ and $\gamma_s = \lfloor 2\gamma_{12} \quad 2\gamma_{13} \rfloor^T$. Shear angles $2\gamma_{12}$ and $2\gamma_{13}$ represent rotations of the cross section about the x_3- and negative x_2 axes, respectively, caused by transverse shear. Strain measures γ_{11}, κ_1, κ_2, and κ_3 are measures of the cross-sectional extension, twist, and the two bending deformations, respectively. As explained in [7], the strain measures from generalized Timoshenko theory differ in meaning from those of classical Euler–Bernoulli theory because of the absence of $2\gamma_{12}$ and $2\gamma_{13}$ from the latter.

The transformation procedure involves kinematic relationships, the 1-D constitutive law, and the 1-D static equilibrium equations. Yu and Hodges [11] show that the kinematic relationships, assuming small values of $2\gamma_{12}$ and $2\gamma_{13}$, between the strain measures are

$$\bar\epsilon = \epsilon + Q\gamma_s' + P\gamma_s \tag{3}$$

where

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad P = \begin{bmatrix} 0 & 0 \\ k_2 & k_3 \\ -k_1 & 0 \\ 0 & -k_1 \end{bmatrix} \tag{4}$$

Having the sectional energy of Eq. (2) implies that the 1-D constitutive law is then

$$\begin{Bmatrix} F_1 \\ M_1 \\ M_2 \\ M_3 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{bmatrix} X & Y \\ Y^T & G \end{bmatrix} \begin{Bmatrix} \epsilon \\ \gamma_s \end{Bmatrix} \tag{5}$$

The linearized 1-D static equilibrium equations, with applied loads set equal to zero (except for the possibility of concentrated loads at the beam ends), are taken from page 79 of [12] and rearranged to read

$$\begin{Bmatrix} F_1 \\ M_1 \\ M_2 \\ M_3 \\ F_2 \\ F_3 \end{Bmatrix}' + \begin{bmatrix} D_3 & D_4 \\ D_2 & D_1 \end{bmatrix} \begin{Bmatrix} F_1 \\ M_1 \\ M_2 \\ M_3 \\ F_2 \\ F_3 \end{Bmatrix} = 0 \tag{6}$$

where

$$D_1 = \begin{bmatrix} 0 & -k_1 \\ k_1 & 0 \end{bmatrix} \qquad D_2 = \begin{bmatrix} k_3 & 0 & 0 & 0 \\ -k_2 & 0 & 0 & 0 \end{bmatrix}$$
$$D_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -k_3 & k_2 \\ 0 & k_3 & 0 & -k_1 \\ 0 & -k_2 & k_1 & 0 \end{bmatrix} \tag{7}$$

and $D_4 = Q - D_2^T$. Note that the generalized Timoshenko strain energy is a quadratic form, so contributions of applied loads do not appear.

The key to the energy transformation is to find expressions for $\bar\epsilon$, $\bar\epsilon'$, and $\bar\epsilon''$ in terms of ϵ and γ_s. It is helpful to introduce matrices $\bar R$, $\bar S$, and $\bar T$ so that

$$\begin{bmatrix} \bar R & \bar S \\ \bar S^T & \bar T \end{bmatrix} \begin{bmatrix} X & Y \\ Y^T & G \end{bmatrix} = \Delta_6 \tag{8}$$

From this definition of the flexibility matrix, one may derive

$$\bar{R} = (X - YG^{-1}Y^T)^{-1} \qquad \bar{S} = -\bar{R}YG^{-1}$$
$$\bar{T} = (\Delta_2 - \bar{S}^T Y)G^{-1} \tag{9}$$

Upon substituting Eq. (5) into Eq. (6) and taking the derivatives, one may express the derivatives of the strain measures as

$$\begin{Bmatrix} \epsilon \\ \gamma_s \end{Bmatrix}' = -[Z]\begin{Bmatrix} \epsilon \\ \gamma_s \end{Bmatrix} \tag{10}$$

where

$$[Z] = \begin{bmatrix} \bar{R} & \bar{S} \\ \bar{S}^T & \bar{T} \end{bmatrix}\left[\begin{bmatrix} X & Y \\ Y^T & G \end{bmatrix}' + \begin{bmatrix} D_3 & D_4 \\ D_2 & D_1 \end{bmatrix}\begin{bmatrix} X & Y \\ Y^T & G \end{bmatrix}\right] \tag{11}$$

By taking derivatives of Eq. (10), along with recursive substitutions, the second and third derivatives of the strain measures are found as

$$\begin{Bmatrix} \epsilon \\ \gamma_s \end{Bmatrix}'' = [Z_2]\begin{Bmatrix} \epsilon \\ \gamma_s \end{Bmatrix}; \qquad \begin{Bmatrix} \epsilon \\ \gamma_s \end{Bmatrix}''' = -[Z_3]\begin{Bmatrix} \epsilon \\ \gamma_s \end{Bmatrix} \tag{12}$$

where

$$[Z_2] = [Z]^2 - [Z]' \qquad [Z_3] = [Z]'' - 2[Z]'[Z] - [Z][Z]' + [Z]^3 \tag{13}$$

Now one may find expressions for matrices X, Y, and G by substituting Eqs. (3) and (10–13) into Eq. (1), and then equating the resulting equation with Eq. (2). The resulting equations involve the unknown stiffness matrices, their derivatives, and known matrices A, B, C, and D. Note that if derivatives of k_1, k_2, and k_3, which implicitly appear in these equations, are nonzero, then it implies the presence of variations in initial twist and initial curvatures.

Two small parameters are identified as a/ℓ and a/R. The small parameters are of the same order and are small in the sense that their cubes are small compared to unity. Terms containing first and second derivatives with respect to x_1 are identified as first- and second-order terms in a/ℓ, and terms multiplied by k_i are identified as first-order in terms of a/R. The ordering presented is essentially that of Starosel'skii [13] and is used to simplify the energy by removing terms of third and higher orders in these parameters.

The resulting equations may be written as

$$X = X_A + X_B + X_C + X_D \qquad Y = Y_A + Y_B + Y_C + Y_D$$
$$G = G_A + G_B + G_C + G_D \tag{14}$$

where the subscript indicates the source of the contribution. For example, Y_A represents the contribution to Y from stiffness matrix A. If only terms through second order in the small parameters are kept, then these individual contributions may be written as

$$\begin{aligned}
X_A &= (\Delta_4 - Q[Z]_{21})^T A(\Delta_4 - Q[Z]_{21}) \\
X_B &= 2(\Delta_4 - Q[Z]_{21})^T B(Q[Z_2]_{21} - [Z]_{11} - P[Z]_{21}) \\
X_C &= (Q[Z_2]_{21} - [Z]_{11} - P[Z]_{21})^T C(Q[Z_2]_{21} - [Z]_{11} - P[Z]_{21}) \\
X_D &= 2(\Delta_4 - Q[Z]_{21})^T D([Z_2]_{11} - Q[Z_3]_{21} + P[Z_2]_{21} - 2P'[Z]_{21})
\end{aligned} \tag{15}$$

$$\begin{aligned}
Y_A &= (\Delta_4 - Q[Z]_{21})^T A(P - Q[Z]_{22}) \\
Y_B &= (\Delta_4 - Q[Z]_{21})^T B(Q[Z_2]_{22} - [Z]_{12} - P[Z]_{22} + P') \\
&\quad + ([Z]_{11} - Q[Z_2]_{21} + P[Z]_{21})^T B^T(Q[Z]_{22} - P) \\
Y_C &= (Q[Z_2]_{21} - [Z]_{11} - P[Z]_{21})^T C(Q[Z_2]_{22} - [Z]_{12} - P[Z]_{22} + P') \\
Y_D &= (\Delta_4 - Q[Z]_{21})^T D([Z_2]_{12} - Q[Z_3]_{22} + P[Z_2]_{22} - 2P'[Z]_{22}) \\
&\quad + (Q[Z_3]_{21} - [Z_2]_{11} - P[Z_2]_{21} + 2P'[Z]_{21})^T D^T(Q[Z]_{22} - P)
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
G_A &= (P - Q[Z]_{22})^T A(P - Q[Z]_{22}) \\
G_B &= 2(P - Q[Z]_{22})^T B(Q[Z_2]_{22} - [Z]_{12} - P[Z]_{22} + P') \\
G_C &= (Q[Z_2]_{22} - [Z]_{12} - P[Z]_{22} + P')^T C(Q[Z_2]_{22} - [Z]_{12} \\
&\quad - P[Z]_{22} + P') \\
G_D &= 2(P - Q[Z]_{22})^T D([Z_2]_{12} - Q[Z_3]_{22} + P[Z_2]_{22} - 2P'[Z]_{22})
\end{aligned} \tag{17}$$

The matrices are partitioned such that subscript 11 refers to the partition occupying rows 1–4 and columns 1–4, subscript 12 refers to the partition occupying rows 1–4 and columns 5–6, subscript 21 refers to the partition occupying rows 5–6 and columns 1–4, and subscript 22 refers to the partition occupying rows 5–6 and columns 5–6. For clarity,

$$[Z] = \begin{bmatrix} [Z]_{11} & [Z]_{12} \\ [Z]_{21} & [Z]_{22} \end{bmatrix} \qquad [Z_2] = \begin{bmatrix} [Z_2]_{11} & [Z_2]_{12} \\ [Z_2]_{21} & [Z_2]_{22} \end{bmatrix}$$
$$[Z_3] = \begin{bmatrix} [Z_3]_{11} & [Z_3]_{12} \\ [Z_3]_{21} & [Z_3]_{22} \end{bmatrix} \tag{18}$$

Explicit expressions for matrices $[Z]'$ and $[Z]''$ are found by simply taking derivatives of $[Z]$. Matrix $[Z]'$ is found to be

$$\begin{aligned}
[Z]' &= \begin{bmatrix} \bar{R} & \bar{S} \\ \bar{S}^T & \bar{T} \end{bmatrix}'\left[\begin{bmatrix} X & Y \\ Y^T & G \end{bmatrix}' + \begin{bmatrix} D_3 & D_4 \\ D_2 & D_1 \end{bmatrix}\begin{bmatrix} X & Y \\ Y^T & G \end{bmatrix}\right] \\
&\quad + \begin{bmatrix} \bar{R} & \bar{S} \\ \bar{S}^T & \bar{T} \end{bmatrix}\left[\begin{bmatrix} X & Y \\ Y^T & G \end{bmatrix}'' + \begin{bmatrix} D_3 & D_4 \\ D_2 & D_1 \end{bmatrix}'\begin{bmatrix} X & Y \\ Y^T & G \end{bmatrix}\right] \\
&\quad + \begin{bmatrix} \bar{R} & \bar{S} \\ \bar{S}^T & \bar{T} \end{bmatrix}\begin{bmatrix} D_3 & D_4 \\ D_2 & D_1 \end{bmatrix}\begin{bmatrix} X & Y \\ Y^T & G \end{bmatrix}'
\end{aligned} \tag{19}$$

$[Z]''$ may be simplified by discarding terms that are higher than second order in terms of the small parameters. The submatrices of $[Z]''$, after discarding most higher-order terms, are then

$$\begin{aligned}
[Z]''_{11} &= \bar{R}''QY^T + 2\bar{R}'Q(Y^T)' + \bar{R}Q(Y^T)'' \\
[Z]''_{12} &= \bar{R}''QG + 2\bar{R}'QG' + \bar{R}QG'' \\
[Z]''_{21} &= (\bar{S}^T)''QY^T + 2(\bar{S}^T)'Q(Y^T)' + \bar{S}^TQ(Y^T)'' \\
[Z]''_{22} &= (\bar{S}^T)''QG + 2(\bar{S}^T)'QG' + \bar{S}^TQG''
\end{aligned} \tag{20}$$

Some higher-order terms, which cannot affect a perturbation solution through second order, are kept so as to keep the simplicity of Eq. (20). In other words, some terms from Eq. (20) implicitly embed both second-order and higher-order terms, but decomposing these terms to keep only their second-order components would significantly increase both the length of Eq. (20) and the effort required for programming. The energy transformation problem is then restated as finding X, Y, and G to satisfy the system of equations given by Eq. (14).

Spanwise nonuniformity effects originate from the appearance of derivative terms. In the energy transformation, these terms are derivatives of the stiffness and flexibility matrices. For beams featuring variations in k_1, k_2, or k_3, nonuniformity effects also appear from derivatives of P.

III. Validation

The energy transformation into generalized Timoshenko form is now applied to the linearly tapered isotropic strip beam, along with relevant information from [3]. Considering only in-plane deformations, the classical 1-D strain measures are

$$\bar{\epsilon} = \lfloor \bar{\gamma}_{11} \quad \bar{\kappa}_3 \rfloor^T \tag{21}$$

while the 1-D strain measures in a generalized Timoshenko model are

$$\epsilon = \lfloor \gamma_{11} \quad \kappa_3 \rfloor^T \qquad \gamma_s = \lfloor 2\gamma_{12} \rfloor \tag{22}$$

For this case, the midplane of the strip is a plane of symmetry in the deformed state so that both κ_2 and $2\gamma_{13}$ are zero. This is required by the isotropy, symmetry of the strip, and the loading considered. The kinematic relationships between the strain measures are still given by Eq. (3), but now with

$$Q = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad P = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{23}$$

The second-order asymptotically correct and generalized Timoshenko sectional strain energies are still given by Eqs. (1) and (2), respectively, but with the modifications that

$$[A] = \begin{bmatrix} A_{11} & A_{14} \\ A_{14} & A_{44} \end{bmatrix} \qquad [B] = \begin{bmatrix} B_{11} & B_{14} \\ B_{41} & B_{44} \end{bmatrix}$$
$$[C] = \begin{bmatrix} C_{11} & C_{14} \\ C_{14} & C_{44} \end{bmatrix} \qquad [D] = \begin{bmatrix} D_{11} & D_{14} \\ D_{41} & D_{44} \end{bmatrix}$$
$$[X] = \begin{bmatrix} X_{11} & X_{14} \\ X_{14} & X_{44} \end{bmatrix} \qquad [Y] = \begin{bmatrix} Y_{11} \\ Y_{41} \end{bmatrix} \qquad [G] = [G_{11}] \tag{24}$$

The 1-D constitutive law of the generalized Timoshenko model is reduced to

$$\begin{Bmatrix} F_1 \\ M_3 \\ F_2 \end{Bmatrix} = \begin{bmatrix} X & Y \\ Y^T & G \end{bmatrix} \begin{Bmatrix} \epsilon \\ \gamma_s \end{Bmatrix} \tag{25}$$

From [3], the analytical formulae for terms in the second-order asymptotically correct stiffness matrices are

$$A_{11} = 2Etb\left[1 - \frac{2}{3}(1+\nu)\tau^2\right]$$
$$A_{44} = \frac{2}{3}Etb^3\left[1 + \frac{1}{3}(30 + 28\nu)\tau^2\right]$$
$$B_{11} = \frac{2}{3}Et\nu\tau b^2$$
$$B_{44} = -\frac{1}{9}Etb^4\tau(36 + 32\nu)$$
$$C_{44} = \frac{8}{15}Etb^5(1+\nu)$$
$$D_{44} = \frac{1}{45}Etb^5(24 + 22\nu)$$
$$A_{14} = B_{14} = B_{41} = C_{14} = D_{14} = D_{41} = 0 \tag{26}$$

while terms of the generalized Timoshenko stiffness matrices are

$$X_{11} = 2Etb\left(1 - \frac{2}{3}\tau^2\right) \qquad X_{44} = \frac{2}{3}Etb^3\left[1 + \frac{(\nu - 48)\nu - 45}{45(\nu+1)}\tau^2\right]$$
$$Y_{41} = \frac{Et(5\nu+3)b^2\tau}{9(\nu+1)} \qquad G_{11} = \frac{5Etb}{6(1+\nu)} \qquad X_{14} = Y_{11} = 0 \tag{27}$$

Note that b is linearly varying such that $\tau = -b'$. Terms involving τ represent nonuniformity effects on the stiffness constants. The equations involved in the transformation procedure into generalized Timoshenko form remain unchanged in their symbolic form, but are reduced in dimension. The formulae, given in Eq. (27), represent the standards to which numerical results from this section must be held.

For the sole purpose of validating the transformation procedure, the reduced system of equations represented by Eq. (14) is solved while making use of two sets of analytical formulae. First, matrices A, B, C, and D, which are derived as part of the second-order asymptotically correct energy, are taken from Eq. (26). Second, derivatives of X, Y, and G are assumed to be known by taking derivatives of Eq. (27). X, Y, and G are then the only unknowns in Eq. (14). For a specified geometry, the unknown matrices are solved iteratively by the MATLAB nonlinear solver command `fsolve`.

Figures 2 and 3 show the extensional stiffness X_{11} and bending stiffness X_{44} as functions of the taper parameter τ, respectively, for $\nu = 0.3$. The results plotted have been normalized by dividing through by the values they would have were $\tau = 0$. Curves labeled Analytical and Calculated are from using Eq. (27) and from solving Eq. (14), respectively. Figure 4 shows the bending-shear coupling Y_{41} as a function of τ for $\nu = 0.3$, $E = 200$ GPa, $t = 0.1$ m, and $b = 3$ m. Note that the energy derived is second-order. Therefore, the

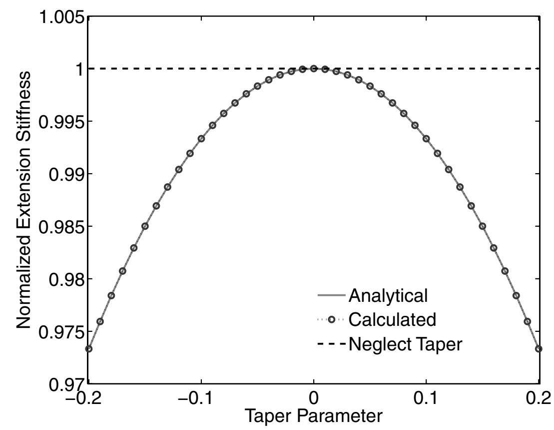

Fig. 2 Generalized Timoshenko extensional stiffness with an isotropic tapered strip.

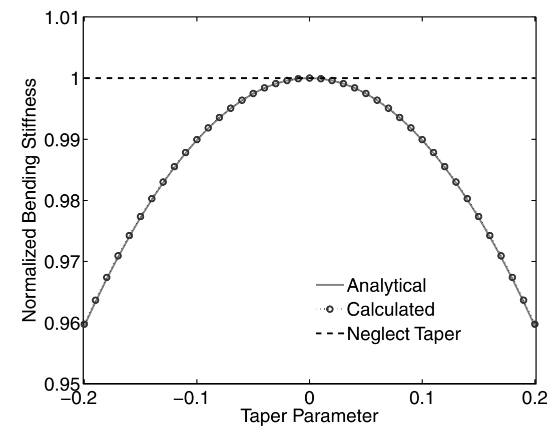

Fig. 3 Generalized Timoshenko bending stiffness about the x_3 axis with an isotropic tapered strip and $\nu = 0.3$.

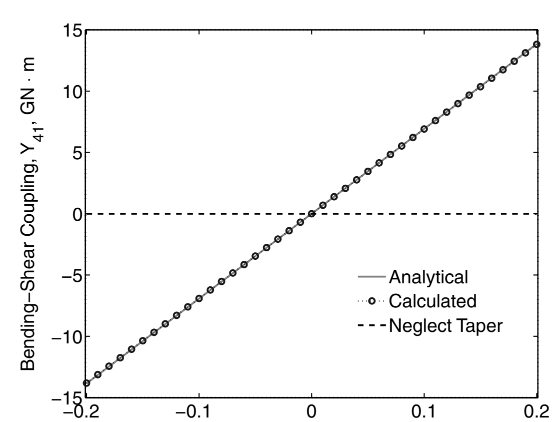

Fig. 4 Generalized Timoshenko bending-shear coupling Y_{41} with an isotropic tapered strip and $\nu = 0.3$.

orders of accuracy of matrices X, Y, and G, are second, first, and zeroth, respectively, because γ_s is a first-order quantity. Formulae given in Eq. (27) are found using a series expansion from which terms of orders higher than the highest order of accuracy have been discarded. To reflect this truncation in the calculated results, its curves are calculated from a quadratic curve-fitting procedure, using only the three values at $\tau = -0.01$, 0.0, and 0.01. The curve fit is justified, because the underlying principle of the asymptotic method is that higher-order corrections are found by an expansion about the zeroth-order solution, which is the prismatic case. For a high level of accuracy to the curve fit, the values of τ are chosen close to the prismatic case. Analytical and calculated results are visibly indistinguishable, so the numerical transformation procedure is a success for this case. The curves labeled Neglect Taper represent calculated stiffnesses from neglecting the taper effect, so its departures from the other curves represent the errors from neglecting the taper effect. The calculated results for X_{14}, Y_{11}, and G_{11}, all quantities that are independent of τ, are correctly predicted and not shown.

Results, discussed from the previous paragraph, validate the system of equations for the transformation procedure to generalized Timoshenko form, but issues still exist. In validating the equations, the Calculated results presumed knowledge of A, B, and the derivatives of X, Y, and G. Ho et al. [9] proposed estimating the derivatives of X, Y, and G by finite difference approximations, but their procedure does not yield sufficiently accurate results [10]. A satisfactory way of determining these presumed quantities remains elusive.

IV. Conclusions

A crucial aspect of the commercialized 2-D cross-sectional analysis tool VABS is the strain-energy transformation from a second-order asymptotically correct theory to the generalized Timoshenko theory. This Technical Note presents details of the transformation with the inclusion of 3-D spanwise nonuniformity effects. The transformation is validated for linearly tapered isotropic strip beams by correlations with a set of previously validated analytical results. This work represents an advancement in the pursuit of providing a sectional analysis tool that is valid for nonuniform beams composed of arbitrary materials and geometries. However, further work is needed in other aspects before such an analysis tool may be realized. Specifically, matrices A and B, from the second-order asymptotically correct theory, and derivatives of the generalized Timoshenko stiffness matrix are currently unknown yet needed quantities to complete the transformation.

Acknowledgments

This research is supported by the U.S. Army Vertical Lift Research Center of Excellence at Georgia Institute of Technology and its affiliate program through subcontract at Utah State University. The Technical Monitor is Michael J. Rutkowski.

References

[1] Timoshenko, S. P., and Goodier, J. N., *Theory of Elasticity*, McGraw–Hill, Maidenhead, England, U.K., 1970.

[2] Krahula, J. L., "Shear Formula for Beams of Variable Cross Section," *AIAA Journal*, Vol. 13, No. 10, Oct. 1975, pp. 1390–1391. doi:10.2514/3.6996

[3] Hodges, D. H., Ho, J. C., and Yu, W., "The Effect of Taper on Section Constants for In-Plane Deformation of an Isotropic Strip," *Journal of Mechanics of Materials and Structures*, Vol. 3, No. 3, Mar. 2008, pp. 425–440. doi:10.2140/jomms.2008.3.425

[4] Berdichevsky, V. L., "Variational-Asymptotic Method of Constructing a Theory of Shells," *Prikladnaya Matematika i Mekhanika*, Vol. 43, No. 4, 1979, pp. 664–687.

[5] Cesnik, C. E. S., and Hodges, D. H., "VABS: A New Concept for Composite Rotor Blade Cross-Sectional Modeling," *Journal of the American Helicopter Society*, Vol. 42, No. 1, Jan. 1997, pp. 27–38. doi:10.4050/JAHS.42.27

[6] Danielson, D. A., and Hodges, D. H., "Nonlinear Beam Kinematics by Decomposition of the Rotation Tensor," *Journal of Applied Mechanics*, Vol. 54, No. 2, Jun. 1987, pp. 258–262. doi:10.1115/1.3173004

[7] Popescu, B., and Hodges, D. H., "On Asymptotically Correct Timoshenko-Like Anisotropic Beam Theory," *International Journal of Solids and Structures*, Vol. 37, No. 3, Jan. 2000, pp. 535–558. doi:10.1016/S0020-7683(99)00020-7

[8] Yu, W., Hodges, D. H., Volovoi, V. V., and Cesnik, C. E. S., "On Timoshenko-Like Modeling of Initially Curved and Twisted Composite Beams," *International Journal of Solids and Structures*, Vol. 39, No. 19, Sept. 2002, pp. 5101–5121. doi:10.1016/S0020-7683(02)00399-2

[9] Ho, J. C., Hodges, D. H., and Yu, W., "Modeling Spanwise Nonuniformity in the Cross-Sectional Analysis of Composite Rotor Blades," *Proceedings of the 65th Annual Forum of the American Helicopter Society*, AHS International, Alexandria, VA, May 2009, pp. 707–726.

[10] Ho, J. C., "Modeling Spanwise Nonuniformity in the Cross-Sectional Analysis of Composite Beams," Ph.D. Dissertation, School of Aerospace Engineering, Georgia Inst. of Technology, Atlanta, Aug. 2009.

[11] Yu, W., and Hodges, D. H., "Generalized Timoshenko Theory of the Variational Asymptotic Beam Sectional Analysis," *Journal of the American Helicopter Society*, Vol. 50, No. 1, Jan. 2005, pp. 46–55. doi:10.4050/1.3092842

[12] Hodges, D. H., *Nonlinear Composite Beam Theory*, Vol. 213, of *Progress in Astronautics and Aeronautics*, AIAA, Reston, VA, 2006.

[13] Starosel'skii, L. A., "On Equations Describing the Vibrations of Curvilinear Elastic Rods," *Prikladnaya Matematika i Mekhanika*, Vol. 24, No. 7, 1979, pp. 584–585.

F. Pai
Associate Editor